

EPHOU-03-008  
 hep-th/0402010  
 February 2004

# Matrix Configurations for Spherical 4-branes and Non-commutative Structures on $S^4$

Ryuichi NAKAYAMA\* and Yusuke SHIMONO†

Division of Physics, Graduate School of Science,  
 Hokkaido University, Sapporo 060-0810, Japan

## Abstract

We present a Matrix theory action and Matrix configurations for spherical 4-branes. The dimension of the representations is given by  $N = 2(2j + 1)$  ( $j = 1/2, 1, 3/2, \dots$ ). The algebra which defines these configurations is not invariant under  $SO(5)$  rotations but under  $SO(3) \otimes SO(2)$ . We also construct a non-commutative product  $\star$  for field theories on  $S^4$  in terms of that on  $S^2$ . An explicit formula of the non-commutative product which corresponds to the  $N = 4$  dim representation of the non-commutative  $S^4$  algebra is worked out. Because we use  $S^2 \otimes S^2$  parametrization of  $S^4$ , our  $S^4$  is doubled and the non-commutative product and functions on  $S^4$  are indeterminate on a great circle ( $S^1$ ) on  $S^4$ . We will however, show that despite this mild singularity it is possible to write down a finite action integral of the non-commutative field theory on  $S^4$ . NS-NS  $B$  field background on  $S^4$  which is associated with our Matrix  $S^4$  configurations is also constructed.

---

\*nakayama@particle.sci.hokudai.ac.jp

†yshimono@particle.sci.hokudai.ac.jp

# 1 Introduction

The fuzzy 4-sphere ( $S^4$ ) was considered in [1] to describe spherical Longitudinal 5-branes (L5-branes) in the context of the Matrix Theory [2][3] and to construct an example of finite 4d field theory[4][5]. By using the  $n$ -fold symmetric tensor product representation of the 4d gamma matrices,  $N = (n+1)(n+2)(n+3)/6$  dimensional Matrix L5-brane configuration was constructed. There is, however, a problem in describing the fluctuating L5-branes. Later it was concluded that ‘fuzzy  $S^4$ ’ is 6 dimensional, *i.e.*, a fuzzy  $S^2$  fibre bundle over a non-associative  $S^4$ . [6][7]

In the above works the assumption of the  $SO(5)$  invariance of the fuzzy  $S^4$  algebra was crucial. What underlies the non-commutativity is, however, the symplectic form or the NS-NS  $B$  field background and there are no rotationally invariant symplectic forms on  $S^m$  except for  $S^2$ . Actually, to construct  $SO(5)$  invariant  $B$  field background it is necessary to introduce an extra internal space and  $SU(2)$  gauge symmetry. [8][9]

In this paper we will propose a new non-commutative  $S^4$  algebra. We do *not* assume  $SO(5)$  invariance of the algebra and do *not* introduce an extra internal space. Our  $S^4$  algebra has  $SO(3) \otimes SO(2)$  symmetry, the subgroup of  $SO(5)$ . It is not a Lie algebra. This algebra has  $N = 2(2j+1)$  dimensional irreducible representations ( $j = 1/2, 1, 3/2, \dots$ ) and can be derived from a Matrix theory action as equations of motion. These representations are given in the form of a tensor product of two representations of  $SO(3)$ . Because  $SO(3)$  is the defining algebra of the non-commutative  $S^2$  [10] and the matrix  $\leftrightarrow$  function correspondence for  $S^2$  is well known [11][12], the non-commutative geometry of  $S^4$  can be realized in terms of the non-commutative structures on two  $S^2$ ’s. We can define the functions and the non-commutative product  $\star$  on  $S^4$  in terms of those on  $S^2$ . The formula for the special case of  $N = 4$  dim representation will be worked out explicitly. Actually, the topologies of  $S^4$  and  $S^2 \otimes S^2$  are different and there appears mild singularity (indeterminateness) on a circle on  $S^4$ . We will, however, show that this singularity is mild enough such that a finite action integral can be written down. This situation is reasonable from the matrix  $\leftrightarrow$  function correspondence: in the Matrix theory there is no singularity in a finite-size matrix. We will also show that  $S^4$  constructed in the  $S^2 \otimes S^2$  parametrization has a twofold structure.

The structure of this paper is as follows. In sec.2 we will define a Matrix theory action for spherical 4-brane and derive the non-commutative  $S^4$  alge-

bra. We will show that the  $SO(5)$  symmetry of  $S^4$  cannot be kept intact in the algebra. We find Matrix configurations with dimension  $N = 2(2j + 1)$  ( $j = 1/2, 1, 3/2, \dots$ ). The corresponding non-commutative  $S^4$  will be called  $(S^4)_j$ . In sec.3 a non-commutative product  $\star$  for non-commutative field theories on  $(S^4)_j$  will be constructed in terms of the non-commutative product  $*$  on  $S^2$ . We point out that our  $S^4$  has a twofold structure. Functions on  $(S^4)_j$  will be also identified. In sec.4 the non-commutative product for  $(S^4)_{1/2}$  will be presented explicitly and in sec.5 we present the NS-NS  $B$  field background corresponding to the Matrix configurations obtained in sec.2. In sec.6 discussions are presented. In appendix A we give the multiplication rule of the functions on  $(S^4)_{1/2}$ . In appendix B the coefficient functions  $L^{(n)}$  which appear in the expression of the non-commutative product  $\star$  on  $(S^4)_{1/2}$  are presented.

Throughout this paper we will let the initial lowercase romans  $a, b, c, \dots$  to take values  $1, 2, 3$  and the middle romans  $i, j, \dots = 4, 5$ , while the capitals  $A, B, \dots$  will run from 1 to 5.

## 2 The Algebra for Non-commutative $S^4$

In this section we will propose the algebra of the coordinates of the non-commutative  $S^4$ . At the beginning we will assume  $SO(5)$  invariance of the algebra. We will, however, shortly find this is not the case. Because the invariant tensors are  $\delta_{AB}$  and  $\epsilon_{ABCDE}$ , the algebra of the coordinates will be given, up to a multiplicative constant, by

$$[\hat{X}^A, \hat{X}^B] = \epsilon_{ABCDE} \hat{X}^C \hat{X}^D \hat{X}^E. \quad (A, B, \dots = 1, 2, \dots, 5) \quad (1)$$

Here  $\hat{X}^A$ 's are finite-dimensional hermitian matrices and  $\epsilon_{ABCDE}$  is the Levi-Civita symbol. One can show that the condition

$$(\hat{X}^A)^2 - C^2 \mathbf{1} = 0 \quad (2)$$

commutes with  $\hat{X}^A$ 's, where  $C$  is a scalar and  $\mathbf{1}$  an identity matrix.

The algebra (1) can be derived as equations of motion from the following action.

$$\int dt \text{Tr} \left\{ \frac{1}{2R_M} (D_0 \hat{X}^A)^2 + \frac{1}{4} ([\hat{X}^A, \hat{X}^B] - \epsilon_{ABCDE} \hat{X}^C \hat{X}^D \hat{X}^E)^2 \right\} \quad (3)$$

Here  $D_0$  is the covariant derivative  $\partial_t + i [\hat{A}_0, \ ]$  and  $R_M$  the radius of the M circle. This may be regarded as the bosonic part of the M-atrrix theory in some

background.<sup>1</sup> Here we omit other bosonic coordinates,  $\hat{X}^6, \dots, \hat{X}^9$ . The potential is positive semi-definite and takes the minimal value when  $\hat{X}^A$  satisfies the algebra (1). The time-independent solution to (1) solves the equation of motion for the action (3).

A representation of (1) in terms of  $4 \times 4$  matrices is given by tensor products of Pauli matrices.<sup>2</sup>

$$\begin{aligned}\hat{X}_0^1 &= \frac{1}{3}\sigma_3 \otimes \sigma_1, & \hat{X}_0^2 &= \frac{1}{3}\sigma_3 \otimes \sigma_2, \\ \hat{X}_0^3 &= \frac{1}{3}\sigma_3 \otimes \sigma_3, & \hat{X}_0^4 &= \frac{1}{3}\sigma_1 \otimes \mathbf{1}_2, \\ \hat{X}_0^5 &= \frac{1}{3}\sigma_2 \otimes \mathbf{1}_2\end{aligned}\tag{4}$$

$\mathbf{1}_2$  stands for  $2 \times 2$  identity matrix. These are four dimensional gamma matrices. It is then natural to try  $N = 2(2j+1)$  dimensional representation by replacing one of the two Pauli matrices by a spin- $j$  representation of  $SO(3)$ ,  $T_{(j)}^a$ :  $\left([T_{(j)}^a, T_{(j)}^b] = i \epsilon_{abc} T_{(j)}^c\right)$ .

$$\begin{aligned}\hat{X}_0^1 &= \frac{2}{3}\sigma_3 \otimes T_{(j)}^1, & \hat{X}_0^2 &= \frac{2}{3}\sigma_3 \otimes T_{(j)}^2, \\ \hat{X}_0^3 &= \frac{2}{3}\sigma_3 \otimes T_{(j)}^3, & \hat{X}_0^4 &= \frac{1}{3}\sigma_1 \otimes \mathbf{1}_{2j+1}, \\ \hat{X}_0^5 &= \frac{1}{3}\sigma_2 \otimes \mathbf{1}_{2j+1}\end{aligned}\tag{5}$$

Indeed, these  $\hat{X}_0^A$ 's satisfy (1) *except* for the following commutator.

$$[\hat{X}^4, \hat{X}^5] = \frac{3}{4j(j+1)} \epsilon_{45CDE} \hat{X}^C \hat{X}^D \hat{X}^E\tag{6}$$

Although this equation breaks the symmetry  $SO(5)$  down to  $SO(3) \otimes SO(2)$ , this algebra is a natural modification of (1). To recover the  $SO(5)$  symmetry we may try rescaling  $\hat{X}_0^a$  ( $a = 1, 2, 3$ ) by a constant  $\alpha$  and  $\hat{X}_0^i$  ( $i = 4, 5$ ) by  $\beta$ , respectively:

$$\begin{aligned}\hat{X}_0^1 &= \frac{2}{3}\alpha \sigma_3 \otimes T_{(j)}^1, & \hat{X}_0^2 &= \frac{2}{3}\alpha \sigma_3 \otimes T_{(j)}^2, \\ \hat{X}_0^3 &= \frac{2}{3}\alpha \sigma_3 \otimes T_{(j)}^3, & \hat{X}_0^4 &= \frac{1}{3}\beta \sigma_1 \otimes \mathbf{1}_{2j+1}, \\ \hat{X}_0^5 &= \frac{1}{3}\beta \sigma_2 \otimes \mathbf{1}_{2j+1}\end{aligned}\tag{7}$$

---

<sup>1</sup>The bosonic part of the action for Matrix theory in the pp-wave background has this form.[13]

<sup>2</sup>We attached a subscript 0 to  $\hat{X}^A$ 's to show that these are particular matrices that satisfy algebra (1).

We then obtain the algebra

$$\begin{aligned}
[\hat{X}^a, \hat{X}^b] &= \frac{\alpha}{\beta^2} \epsilon_{abcij} \left( \hat{X}^c \hat{X}^i \hat{X}^j - \hat{X}^i \hat{X}^c \hat{X}^j + \hat{X}^i \hat{X}^j \hat{X}^c \right), \\
[\hat{X}^a, \hat{X}^i] &= \frac{1}{\alpha} \epsilon_{aibcj} \left( \hat{X}^b \hat{X}^c \hat{X}^j - \hat{X}^b \hat{X}^j \hat{X}^c + \hat{X}^j \hat{X}^b \hat{X}^c \right), \\
[\hat{X}^4, \hat{X}^5] &= \frac{3\beta^2}{4j(j+1)\alpha^3} \epsilon_{45abc} \hat{X}^a \hat{X}^b \hat{X}^c.
\end{aligned} \tag{8}$$

(Throughout this paper  $a, b, .. = 1, 2, 3$  and  $i, j, .. = 4, 5$ , while  $A, B, .. = 1, 2, ..., 5$ .) When we try to equate the prefactors on the RHS, we find that the equations  $\alpha/\beta^2 = 1/\alpha = 3\beta^2/(4j(j+1)\alpha^3)$  for  $\alpha, \beta$  do not have a solution except for the case  $j = 1/2$ . Hence rescaling of  $\hat{X}_0^A$ 's does not save the symmetry; we conclude that we must abandon  $SO(5)$  symmetry.

Although  $\alpha^{-2} (\hat{X}^a)^2 + \beta^{-2} (\hat{X}^i)^2$  does *not* commute with  $\hat{X}^A$  in the algebra (8), the configuration (7) *does* satisfy the following equation.

$$\alpha^{-2} (\hat{X}_0^a)^2 + \beta^{-2} (\hat{X}_0^i)^2 = \frac{2}{9} (2j+1)^2 \mathbf{1} \tag{9}$$

Since (8) is not a Lie algebra, this is possible. These matrices do not commute with each other and are not decoupled; the matrices (7) may be called a Matrix  $S^4$  configuration.<sup>3</sup> We will denote this configuration as  $(S^4)_j$ . Its dimension  $N = 2(2j+1)$  is extending to infinity. The structures of  $\hat{X}_0^{4,5}$  in (7) are quite simple compared to those of  $\hat{X}_0^{1,2,3}$ . Especially, the eigenvalues of  $\hat{X}_0^{4,5}$  are  $\pm\beta$ , while those of  $\hat{X}_0^{1,2,3}$  are randomly distributed over the range between  $-(2j/3)\alpha$  and  $(2j/3)\alpha$ . In sec.6 we will consider more general matrices. For these the eigenvalues of  $\hat{X}_0^{1,2,3,4,5}$  are all randomly distributed.

We expect that when the size of the matrices becomes large, the matrices  $\hat{X}_0^A$  will commute and the classical geometry of  $S^4$  will be recovered as in the case of fuzzy  $S^2$ . To show that this is really the case we must replace the Pauli matrices in the tensor products in (7) to matrices of an arbitrary representation (spin  $j'$ ). In sec.6 we will discuss the commutative limit.

Finally, the Matrix theory action for spherical 4-branes is given by

$$\begin{aligned}
S &= \int dt \text{Tr} \left\{ \frac{1}{2R_M} (D_0 \hat{X}^A)^2 \right. \\
&\quad + \frac{1}{4} \left( [\hat{X}^a, \hat{X}^b] - \frac{\alpha}{\beta^2} \epsilon_{abCDE} \hat{X}^C \hat{X}^D \hat{X}^E \right)^2 \\
&\quad \left. + \frac{1}{2} \left( [\hat{X}^a, \hat{X}^i] - \frac{1}{\alpha} \epsilon_{aiCDE} \hat{X}^C \hat{X}^D \hat{X}^E \right)^2 \right\}
\end{aligned}$$

---

<sup>3</sup>For  $\alpha \neq \beta$  it may be more pertinent to call this an ellipsoid. See footnote 4.

$$+\frac{1}{2}\left([\hat{X}^4, \hat{X}^5] - \frac{3\beta^2}{4j(j+1)\alpha^3} \epsilon_{45CDE} \hat{X}^C \hat{X}^D \hat{X}^E\right)^2\Big\}. \quad (10)$$

By shifting  $\hat{X}^A \rightarrow \hat{X}_0^A + \hat{A}^A$  in this action, where  $\hat{X}_0^A$  is the Matrix  $S^4$  configuration (7), we obtain a matrix form of the action of the non-commutative gauge theory on  $S^4$ .

### 3 Non-commutative Product on $(S^4)_j$

In this section we will construct a non-commutative product on  $S^4$  corresponding to the Matrix configuration (7) in terms of the product on  $S^2$ .

#### 3.1 Non-commutative $S^2$

Two dimensional fuzzy sphere is defined by the  $SO(3)$  algebra[10]

$$[\hat{X}^a, \hat{X}^b] = i\epsilon_{abc} \hat{X}^c \quad (a, b, .. = 1, 2, 3). \quad (11)$$

This algebra can be realized in the space of functions on the sphere, where  $\hat{X}^a$  is represented by a variable  $x^a$  and the multiplication rule for the functions on the sphere is defined by a noncommutative product  $*$ .<sup>4</sup>

$$f(x) * g(x) = f(x) g(x) + \sum_{m=1}^{\infty} \lambda^m C_m(\lambda) J_{a_1 b_1}(x) \cdots J_{a_m b_m}(x) \times \partial_{a_1} \cdots \partial_{a_m} f(x) \partial_{b_1} \cdots \partial_{b_m} g(x) \quad (12)$$

Here  $f(x)$  and  $g(x)$  are functions of  $x^a/r$ . ( $r = \sqrt{(x^a)^2}$ )  $C_m(\lambda)$  and  $J_{ab}(x)$  are defined by

$$C_m(\lambda) = \frac{\lambda^m}{m!(1-\lambda)(1-2\lambda)\cdots(1-(m-1)\lambda)}, \quad (13)$$

$$J_{ab}(x) = r^2 \delta_{ab} - x^a x^b + i r \epsilon_{abc} x^c. \quad (14)$$

By using this product we obtain the star-commutator.

$$[x^a, x^b]_* \equiv x^a * x^b - x^b * x^a = 2i \lambda r \epsilon_{abc} x^c \quad (15)$$

By comparing (11) and (15) we find the correspondence  $\hat{X}^a \leftrightarrow \frac{x^a}{2\lambda r}$ . For spin- $j$  representation of  $SO(3)$  the quadratic Casimir operator takes a value  $j(j+1)$  ( $j = 1/2, 1, 3/2, \dots$ ). By using

$$x^a * x^a = (1 + 2\lambda)r^2 \quad (16)$$

---

<sup>4</sup>We will use a symbol  $*$  for the non-commutative product on  $S^2$  and  $\star$  for that on  $S^4$ .

we obtain two values  $\lambda = 1/(2j)$  and  $\lambda = -1/(2j+2)$ .

The first value  $\lambda = 1/(2j)$  corresponds to a finite set of functions on the sphere. The set of the polynomials of  $x^a/r$ , spherical harmonics, up to order  $2j$  are closed under the multiplication defined by (12) and the number of spherical harmonics is given by  $\sum_{\ell=0}^{2j} (2\ell+1) = (2j+1)^2$ . This is equal to the number of independent components of  $(2j+1) \times (2j+1)$  hermitian matrix. In this case the summation over  $m$  in (12) must be terminated at  $m = 2j$ .

For the second value  $\lambda = -1/(2j+2)$  the spherical harmonics to arbitrary orders are produced under multiplication of  $x^a/r$ 's and the number of independent functions is infinite.

In this paper we will restrict our attention to a finite set of functions on the non-commutative  $S^4$  and henceforth concentrate on the first value  $\lambda = 1/(2j)$ . In this case we find the matrix  $\leftrightarrow$  function correspondence, the correspondence between the generator in the spin  $j$  representation,  $\hat{X}^a = T_{(j)}^a$ , and  $x^a$ .

$$T_{(j)}^a \leftrightarrow j \frac{x^a}{r} \quad (17)$$

Both sides satisfy the same  $SO(3)$  algebra.

### 3.2 $S^2 \otimes S^2$ parametrization of $S^4$

We will now apply the correspondence (17) to the  $S^4$  configuration (7). These matrices are written as tensor products of two matrices, whose dimensions are 2 and  $2j+1$ , respectively.  $2 \times 2$  matrices are spanned by  $\mathbf{1}$  and  $\sigma_a$ , while  $(2j+1) \times (2j+1)$  matrices are spanned by  $\mathbf{1}$  and the products of  $T_{(j)}^a$ 's. By matrix  $\leftrightarrow$  function correspondence these matrices are realized by polynomials on  $S^2$ . Then we have to introduce two  $S^2$ 's. The coordinates of each  $S^2$  are denoted by  $(x^1, x^2, x^3)$  and  $(y^1, y^2, y^3)$ , respectively. The radii of the two  $S^2$ 's are  $r = \sqrt{(x^a)^2}$  and  $\rho = \sqrt{(y^a)^2}$ . Then in (7) we replace  $\frac{1}{2}\sigma_a$  by  $x^a/2r$  and  $T_{(j)}^a$  by  $j y^a/\rho$ .

$$\begin{aligned} \hat{X}_0^1 \leftrightarrow X^1 &= R \frac{x^3 y^1}{r \rho}, & \hat{X}_0^2 \leftrightarrow X^2 &= R \frac{x^3 y^2}{r \rho}, \\ \hat{X}_0^3 \leftrightarrow X^3 &= R \frac{x^3 y^3}{r \rho}, & \hat{X}_0^4 \leftrightarrow X^4 &= R \frac{x^1}{r}, \\ \hat{X}_0^5 \leftrightarrow X^5 &= R \frac{x^2}{r} \end{aligned} \quad (18)$$

Here we set  $\alpha = 3R/2j$  and  $\beta = 3R$ . These relations give a transformation from  $\left(\frac{x^a}{r}, \frac{y^b}{\rho}\right)$  to  $\left(\frac{X^A}{R}\right)$ . Because we can check that<sup>5</sup>

$$\sum_{A=1}^5 (X^A)^2 = R^2, \quad (\text{ordinary product}) \quad (19)$$

(18) yields a map from  $S^2 \otimes S^2$  to  $S^4$ . In fact this map gives a double cover of  $S^4$ . Actually, two points on  $S^2 \otimes S^2$ ,

$$P = ((x^1, x^2, x^3)/r, (y^1, y^2, y^3)/\rho) \text{ and } P' = ((x^1, x^2, -x^3)/r, (-y^1, -y^2, -y^3)/\rho)$$

map onto a same point on  $S^4$ . Therefore we can divide the first  $S^2$  into the upper and lower hemispheres,  $S_+^2$  ( $x^3 \geq 0$ ) and  $S_-^2$  ( $x^3 \leq 0$ ), each corresponding to a single  $S^4$ . The boundary points ( $x^3 = 0$ ), which constitute  $S^1 \otimes S^2$ , will be mapped onto a great circle ( $S^1$ ),

$$\mathcal{C} = \{(X^1, X^2, X^3, X^4, X^5) | X^1 = X^2 = X^3 = 0, (X^4)^2 + (X^5)^2 = R^2\}. \quad (20)$$

Consequently, the inverse map  $S^4 \rightarrow S_+^2 \otimes S^2$ ,

$$\begin{aligned} x^1(X) &= r X^4/R, & x^2(X) &= r X^5/R, & x^3(X) &= r D(X)/R, \\ y^a(X) &= \rho X^a/D(X), \end{aligned} \quad (21)$$

is multi-valued (indeterminate) on the circle. The image of each point on  $\mathcal{C}$  is  $S^2$ . Here  $D(X)$  is defined by

$$D(X) \equiv \sqrt{(X^1)^2 + (X^2)^2 + (X^3)^2}. \quad (22)$$

The other inverse map  $S^4 \rightarrow S_-^2 \otimes S^2$ , which is obtained from (21) by the replacement  $D(X) \rightarrow -D(X)$ , is also indeterminate on  $\mathcal{C}$ .

The above analysis shows that in the  $S^2 \otimes S^2$  parametrization the manifold  $S^4$  which corresponds to the Matrix configuration (7) is doubled. The two  $S^4$ 's are attached on  $\mathcal{C}$ . The non-commutative structures on each  $S^4$  are related by  $D(X) \leftrightarrow -D(X)$ . This fact suggests us to assign distinct signs to  $D(X)$  on each sheet of the doubled  $S^4$ :

$$D(X) \equiv \begin{cases} +\sqrt{(X^a)^2} & \text{on the first } S^4 \\ -\sqrt{(X^a)^2} & \text{on the second } S^4 \end{cases} \quad (23)$$

---

<sup>5</sup>In this case (9) will represent an ellipsoid for  $j \neq 1/2$ . If we made a different choice for  $\alpha$  and  $\beta$ , the manifold described by the coordinate  $X^A$  would in turn be deformed into an ellipsoid. For the discussion of this paper this choice of  $\alpha$  and  $\beta$  is just for convenience.

### 3.3 Non-commutative Product

Let us now construct the noncommutative product on  $(S^4)_j$ . Because a function on  $S^2 \otimes S^2$  does not depend on  $r$  and  $\rho$ , this function does not depend on  $R$ , either, and this can also be regarded as that on  $S^4$ . To write down the non-commutative product on  $S^4$  we must regard  $R$  as a function of  $x^a$  and  $y^a$ . To keep  $SO(3)$  symmetry of  $S^2$ 's intact we will take  $R = R(r, \rho)$ . Then (21) can also be regarded as a transformation of six variables  $(x^a, y^b) \rightarrow (X^A, R)$ . By combining the non-commutative products (12) for the two  $S^2$ 's we define the noncommutative product of functions,  $F(X)$  and  $G(X)$ , on  $(S^4)_j$ .

$$\begin{aligned}
& F(X) \star G(X) \\
&= F(X)G(X) + (r^2\delta_{ab} - x^a x^b + i r \epsilon_{abc} x^c) \frac{\partial F(X)}{\partial x^a} \frac{\partial G(X)}{\partial x^b} \\
&\quad + \lambda(\rho^2\delta_{ab} - y^a y^b + i \rho \epsilon_{abc} y^c) \frac{\partial F(X)}{\partial y^a} \frac{\partial G(X)}{\partial y^b} \\
&\quad + \lambda(r^2\delta_{ab} - x^a x^b + i r \epsilon_{abc} x^c)(\rho^2\delta_{de} - y^d y^e + i \rho \epsilon_{def} y^f) \frac{\partial^2 F(X)}{\partial x^a \partial y^d} \frac{\partial^2 G(X)}{\partial x^b \partial y^e} \\
&\quad + \dots
\end{aligned} \tag{24}$$

Here  $\lambda = 1/2j$  and terms with higher orders of  $\lambda$  are not written explicitly. This is still expressed in terms of  $x^a$ ,  $y^a$ , not by  $X^A$ , and implicit. Later we will write down the product for the  $j = 1/2$  case more explicitly.

### 3.4 Functions on Non-commutative $(S^4)_j$

The functions on  $(S^4)_j$  are given by the products of the functions on the two  $S^2$ 's. Functions on the first  $S^2$  are spanned by 1 and  $\{x^a/r, (a = 1, 2, 3)\}$ . Those on the second  $S^2$  are spanned by 1 and  $\{y^{a_1} \dots y^{a_n}/\rho^n, (1 \leq n \leq 2j, a_1, \dots, a_n = 1, 2, 3)\}$ . Then those on  $(S^4)_j$  are given by linear combinations of

$$1, \quad x^a/r, \quad y^{a_1} \dots y^{a_n}/\rho^n, \quad x^a y^{a_1} \dots y^{a_n}/r \rho^n. \tag{25}$$

These functions can be expressed in terms of  $X^A$ 's by the transformation (21).

$$\begin{aligned}
& 1, \quad X^i/R, \quad D(X)/R, \quad X^{a_1} \dots X^{a_n}/D(X)^n, \\
& X^i X^{a_1} \dots X^{a_n}/RD(X)^n, \quad X^i X^{a_1} \dots X^{a_n}/RD(X)^n, \\
& X^{a_1} \dots X^{a_n}/RD(X)^{n-1}
\end{aligned} \tag{26}$$

The number of independent functions is  $4 \cdot (2j+1)^2 = \{2(2j+1)\}^2$ , which is a square of an integer. This is the reason why the functions on  $(S^4)_j$  correspond

to  $2(2j+1) \times 2(2j+1)$  hermitian matrices. All the functions are, however, not polynomials.

The spherical harmonics  $Y_{\ell_1 \ell_2 \ell_3 m}$  on  $S^4$  are labeled by integers  $\ell_1 (= 0, 1, \dots)$ ,  $\ell_2 (= 0, 1, \dots, \ell_1)$ ,  $\ell_3 (= 0, 1, \dots, \ell_2)$ ,  $m (= 0, \pm 1, \dots, \pm \ell_3)$ . These are  $\ell_1$ -th order polynomials and the total number of polynomials up to order  $\ell$  is given by

$$\sum_{\ell_1=0}^{\ell} \sum_{\ell_2=0}^{\ell_1} \sum_{\ell_3=0}^{\ell_2} (2\ell_3 + 1) = \frac{1}{12} (\ell + 1) (\ell + 2)^2 (\ell + 3). \quad (27)$$

This is not a square of an integer. With any choice of  $\ell$  (27) cannot be made equal to  $\{2(2j+1)\}^2$ , the number of independent functions on  $S^4$ . Indeed this fact makes it difficult to establish an isomorphism between matrices and polynomials on  $S^4$  equipped with an associative multiplication rule. [6][7]

Our construction avoids this problem by introducing non-polynomial functions in (26). These functions are, however, not well-defined at all points on  $S^4$ . The map  $S^4 \rightarrow S^2_+ \otimes S^2$  (21) is multi-valued on the circle (20) and the functions (26) are also indeterminate. Actually these functions are defined on  $S^2 \otimes S^2$  and such a singularity is anticipated. In ordinary commutative field theory such a singularity of the fields is not allowed. Derivatives of the fields will become more singular. On the other hand in non-commutative field theory this singularity is, however, not serious. First of all this discontinuity is finite and the functions (26) are integrable on  $S^4$ . In addition, in non-commutative field theories the derivative of a function  $F(X)$  is given by the commutator  $\nabla_A F(X) = \frac{i}{R} [X^A, F(X)]_\star = \frac{i}{R} (X^A \star F(X) - F(X) \star X^A)$ , which can be expressed as a linear combination of the functions (26). Because the non-commutative product contains the derivatives of functions, each term in the product may be singular on the circle. Let us, however, define the non-commutative product of the functions on the circle by a limit from outside the circle. In this case, when the product  $\star$  is properly constructed, it is arranged such that those singularities cancel out completely to leave only the indeterminateness on  $\mathcal{C}$ . Therefore in spite of the indeterminateness of the functions on the circle (20) the product and derivatives of functions are ‘well-defined’ and we can write down a finite action integral. This is sufficient for our purpose.

This point is also evident from the matrix  $\leftrightarrow$  function correspondence: the Matrix configuration (7) does not have any singularity. The trace of a finite-size matrix is finite.

We conclude that we can construct non-commutative field theories on  $S^4$  at the expense of the indeterminateness of functions on the circle (20). It still,

however, remains a possibility that a different method other than our  $S^2 \otimes S^2$  parametrization might lead to non-singular, non-commutative product and functions.

To complete the matrix-function correspondence we need to define the integration measure over  $S^4$ . This is again induced by the integration measures over the two  $S^2$ 's, which are given by  $\int dx^1 dx^2/rx^3$  and  $\int dy^1 dy^2/\rho y^3$ , respectively. By using (21) we define the integration measure over  $S^4$  by

$$\int_{S^4} d^4X \equiv \int \frac{dx^1 dx^2}{r x^3} \frac{dy^1 dy^2}{\rho y^3} = \int \frac{dX^1 dX^2 dX^4 dX^5}{R X^3 D(X)^2}. \quad (28)$$

This does not coincide with the ordinary  $SO(5)$  invariant measure.

## 4 Non-Commutative Product On $(S^4)_{j=1/2}$

We will work out the explicit form of the non-commutative product for the  $j = 1/2$  case. Although it is in principle possible to change variables from  $(x^a, y^a)$  to  $X^A$  in (24), we found it easier to construct the product which reproduces the multiplication rule of the functions (26). In the  $j = 1/2$  case we have the following functions.

$$\begin{aligned} 1, \quad X^i/R &= x^{i-3}/r, \quad D(X)/R = x^3/r, \quad X^a/D(X) = y^a/\rho, \\ X^a/R &= x^3 y^a/r\rho, \quad X^i X^a/RD(X) = x^{i-3} y^a/r\rho \end{aligned} \quad (29)$$

By using the multiplication rule of functions on  $S^2$ ,

$$\begin{aligned} 1 * 1 &= 1, \quad 1 * \frac{x^a}{r} = \frac{x^a}{r} * 1 = \frac{x^a}{r}, \\ \frac{x^a}{r} * \frac{x^b}{r} &= \delta_{ab} + i\epsilon_{abc} \frac{x^c}{r} \end{aligned} \quad (30)$$

and the similar equations for  $y^a/\rho$ , we obtain the multiplication rule of the functions (29). This is presented in Appendix A.

Because the non-commutative product for  $S^2$  in the  $j = 1/2$  representation contains at most first order derivatives of functions, we find that the product for  $(S^4)_j$  contains at most second order derivatives of functions. So we can assume the following form for the product.

$$\begin{aligned} F(X) \star G(X) &= FG \\ &+ L_{a,b}^{(1)} \frac{\partial F}{\partial X^a} \frac{\partial G}{\partial X^b} + L_{i,j}^{(2)} \frac{\partial F}{\partial X^i} \frac{\partial G}{\partial X^j} \end{aligned}$$

$$\begin{aligned}
& + L_{a,i}^{(3)} \frac{\partial F}{\partial X^a} \frac{\partial G}{\partial X^i} + L_{i,a}^{(4)} \frac{\partial F}{\partial X^i} \frac{\partial G}{\partial X^a} \\
& + L_{ab,c}^{(5)} \frac{\partial^2 F}{\partial X^a \partial X^b} \frac{\partial G}{\partial X^c} + L_{a,bc}^{(6)} \frac{\partial F}{\partial X^a} \frac{\partial^2 G}{\partial X^b \partial X^c} \\
& + L_{ab,i}^{(7)} \frac{\partial^2 F}{\partial X^a \partial X^b} \frac{\partial G}{\partial X^i} + L_{a,bi}^{(8)} \frac{\partial F}{\partial X^a} \frac{\partial^2 G}{\partial X^b \partial X^i} \\
& + L_{ai,b}^{(9)} \frac{\partial^2 F}{\partial X^a \partial X^i} \frac{\partial G}{\partial X^b} + L_{i,ab}^{(10)} \frac{\partial F}{\partial X^i} \frac{\partial^2 G}{\partial X^a \partial X^b} \\
& + L_{ai,j}^{(11)} \frac{\partial^2 F}{\partial X^a \partial X^i} \frac{\partial G}{\partial X^j} + L_{a,ij}^{(12)} \frac{\partial F}{\partial X^a} \frac{\partial^2 G}{\partial X^i \partial X^j} \\
& + L_{i,aj}^{(13)} \frac{\partial F}{\partial X^i} \frac{\partial G}{\partial X^a \partial X^j} + L_{ij,a}^{(14)} \frac{\partial^2 F}{\partial X^i \partial X^j} \frac{\partial G}{\partial X^a} \\
& + L_{ij,k}^{(15)} \frac{\partial^2 F}{\partial X^i \partial X^j} \frac{\partial G}{\partial X^k} + L_{i,jk}^{(16)} \frac{\partial F}{\partial X^i} \frac{\partial^2 G}{\partial X^j \partial X^k} \\
& + L_{ab,cd}^{(17)} \frac{\partial^2 F}{\partial X^a \partial X^b} \frac{\partial^2 G}{\partial X^c \partial X^d} + L_{ij,kl}^{(18)} \frac{\partial^2 F}{\partial X^i \partial X^j} \frac{\partial^2 G}{\partial X^k \partial X^l} \\
& + L_{ab,ci}^{(19)} \frac{\partial^2 F}{\partial X^a \partial X^b} \frac{\partial^2 G}{\partial X^c \partial X^i} + L_{ai,bc}^{(20)} \frac{\partial^2 F}{\partial X^a \partial X^i} \frac{\partial^2 G}{\partial X^b \partial X^c} \\
& + L_{ab,ij}^{(21)} \frac{\partial^2 F}{\partial X^a \partial X^b} \frac{\partial^2 G}{\partial X^i \partial X^j} + L_{ij,ab}^{(22)} \frac{\partial^2 F}{\partial X^i \partial X^j} \frac{\partial^2 G}{\partial X^a \partial X^b} \\
& + L_{ai,bj}^{(23)} \frac{\partial^2 F}{\partial X^a \partial X^i} \frac{\partial^2 G}{\partial X^b \partial X^j} \\
& + L_{ai,jk}^{(24)} \frac{\partial^2 F}{\partial X^a \partial X^i} \frac{\partial^2 G}{\partial X^j \partial X^k} + L_{ij,ak}^{(25)} \frac{\partial^2 F}{\partial X^i \partial X^j} \frac{\partial^2 G}{\partial X^a \partial X^k} \quad (31)
\end{aligned}$$

Here  $F(X)$  and  $G(X)$  are arbitrary linear combinations of (29).  $L^{(n)}$  are suitable functions of  $X^A$ . These functions  $L^{(n)}$  must be determined in such a way that (51) is reproduced. In addition we must also require that  $\sqrt{(X^A)^2} = R$  is a constant.

$$f\left(\sqrt{(X^A)^2}\right) \star G(X) = G(X) \star f\left(\sqrt{(X^A)^2}\right) = f\left(\sqrt{(X^A)^2}\right) G(X), \quad (32)$$

where  $f(R)$  is an arbitrary function of  $R$ , and  $G(X)$  a linear combination of (29). Because the non-commutative product of functions contains up to second order derivatives, it is sufficient to take  $f(R) = R, R^2$ .

There is ambiguity in the structure of  $L^{(n)}$ . The origin of this ambiguity lies in the arbitrariness of the  $r, \rho$  dependence of  $R(r, \rho)$ . One of the simplest solutions is presented in Appendix B.

In the non-commutative space the derivatives of the fields are defined in terms of the star commutator with  $X^A$ .

$$\nabla_A G(X) \equiv \frac{i}{R} [X^A, G(X)]_\star \quad (33)$$

Explicitly for  $(S^4)_{1/2}$ , by using (31), (52) we obtain

$$\begin{aligned}
\nabla_a G(X) \equiv & -2 \frac{R}{D(X)} \epsilon_{abc} X^c \frac{\partial G}{\partial X^b} - 2 \frac{1}{D(X)} \epsilon_{ij} X^j X^a \frac{\partial G}{\partial X^i} \\
& - \frac{R^2 - D(X)^2}{RD(X)} (\epsilon_{acd} X^b + \epsilon_{bcd} X^a) X^d \frac{\partial^2 G}{\partial X^b \partial X^c} \\
& + \frac{2}{R} D(X) \epsilon_{abc} X^c X^i \frac{\partial^2 G}{\partial X^b \partial X^i} \\
& - 2i \epsilon_{ij} \epsilon_{abc} X^j X^c \frac{\partial^2 G}{\partial X^b \partial X^i} \\
& + \frac{1}{D(X)} (\epsilon_{ik} X^j + \epsilon_{jk} X^i) X^k X^a \frac{\partial^2 G}{\partial X^i \partial X^j}, \tag{34}
\end{aligned}$$

$$\begin{aligned}
\nabla_i G(X) \equiv & -2D(X) \epsilon_{ij} \frac{\partial G}{\partial X^j} - 2 \frac{1}{D(X)} \epsilon_{ij} X^j X^a \frac{\partial G}{\partial X^a} \\
& - 2 \frac{1}{D(X)} \epsilon_{ij} X^j X^a X^b \frac{\partial^2 G}{\partial X^a \partial X^b} - 2D(X) X^a \epsilon_{ij} \frac{\partial^2 G}{\partial X^a \partial X^j} \\
& - i \frac{D(X)^2}{2R} (\epsilon_{ij} \epsilon_{kl} + \epsilon_{ik} \epsilon_{jl}) X^l \frac{\partial^2 G}{\partial X^j \partial X^k} \tag{35}
\end{aligned}$$

The field strength  $F_{AB}(X)$  of the non-commutative gauge field  $A_A(X)$  is defined by replacing the matrix  $\hat{X}^A$  in

$$\frac{i}{R^2} \left( [\hat{X}^A, \hat{X}^B] - \frac{1}{3R} \epsilon_{ABCDE} \hat{X}^C \hat{X}^D \hat{X}^E \right), \tag{36}$$

which appears in the action (10), by  $X^A + R A_A$ , and the matrix multiplication by  $\star$ . Here we write down only the expression for the  $j = 1/2$  case. We obtain, by setting  $\alpha = \beta = 3R$ ,

$$\begin{aligned}
F_{AB}(X) = & \nabla_A A_B - \nabla_B A_A + i [A_A, A_B]_\star - \frac{i}{3} \epsilon_{ABCDE} A_C \star A_D \star A_E \\
& - \frac{i}{3R^2} \epsilon_{ABCDE} (X^C \star X^D \star A_E + X^C \star A_D \star X^E + A_C \star X^D \star X^E) \\
& - \frac{i}{3R} \epsilon_{ABCDE} (X^C \star A^D \star A_E + A^C \star X_D \star A^E + A_C \star A^D \star X^E). \tag{37}
\end{aligned}$$

The action integral is given by the usual form.

$$S_{\text{gauge theory}} = \int dt \int_{S^4} d^4 X \left( \frac{R^2}{2R_M} D_0 A_A \star D_0 A_A - \frac{R^4}{4} F_{AB} \star F_{AB} \right) \tag{38}$$

## 5 $B$ Field Background

Although non-commutative geometry is not always related to the NS-NS  $B$  field background[14], the relation between the  $B$  field and the non-commutativity is well established.[15] When the  $B$  field two-form is closed,  $dB = 0$ , then the inverse matrix  $\alpha^{AB}$  ( $B_{AB} \alpha^{BC} = \delta_A^C$ ) defines a Poisson bracket,  $\{F, G\}_{PB} = \frac{1}{2} \alpha^{AB} \partial_A F \partial_B G$ , and one can construct an associative, non-commutative product  $F \star G$  by means of perturbation in  $\alpha$  and its derivatives.[16] In what follows we will construct a  $B$  field background (or symplectic form) on  $S^4$ .

Let us introduce an  $S^2 \otimes S^2$  parametrization (18) of  $S^4$ .

$$\begin{aligned} X^1 &= R \cos \theta_1 \sin \theta_2 \cos \varphi_2, & X^2 &= R \cos \theta_1 \sin \theta_2 \sin \varphi_2, \\ X^3 &= R \cos \theta_1 \cos \theta_2, & X^4 &= R \sin \theta_1 \cos \varphi_1, \\ X^5 &= R \sin \theta_1 \sin \varphi_1 \end{aligned} \tag{39}$$

Here  $(\theta_1, \varphi_1)$  and  $(\theta_2, \varphi_2)$  are polar coordinates of two unit spheres ( $0 \leq \theta_i \leq \pi$ ,  $0 \leq \varphi_i \leq 2\pi$ ) corresponding to  $x^a$  and  $y^a$ , respectively. We can check that  $X^A$ 's satisfy  $(X^A)^2 = R^2$ .

This parametrization gives a double cover of  $S^4$ . The points  $P = (\theta_1, \varphi_1, \theta_2, \varphi_2)$  and  $P' = (\pi - \theta_1, \varphi_1, \pi - \theta_2, \varphi_2 + \pi)$  are mapped onto a same point on  $S^4$ . Hence we must divide the first  $S^2$  into an upper hemisphere,  $S_+^2$  ( $0 \leq \theta_1 \leq \pi/2$ ) and a lower one,  $S_-^2$  ( $\pi/2 \leq \theta_1 \leq \pi$ ). We must also be careful with the boundary of the hemispheres ( $\theta_1 = \pi/2$ ,  $0 \leq \varphi_1 \leq 2\pi$ ). For any values of  $\theta_2, \varphi_2$  this is mapped onto a circle (20).

The  $B$  field background which has the maximal symmetry of the two spheres is given by

$$B \equiv \frac{n_1}{2} \sin \theta_1 d\theta_1 \wedge d\varphi_1 + \frac{n_2}{2} \sin \theta_2 d\theta_2 \wedge d\varphi_2. \tag{40}$$

Here the fluxes are quantized as usual and  $n_1, n_2$  are integers. This two-form is closed. Now let us remember that for  $\theta_1 = \pi/2$  the coordinates  $\theta_2$  and  $\varphi_2$  become redundant in (39). Therefore actually, the  $B$  field (40) is not defined at all points on  $S^4$ . In terms of  $X^A$  we obtain

$$\begin{aligned} B &= \frac{n_2}{4D(X)^3} \epsilon_{abc} X^a dX^b \wedge dX^c + \frac{n_1}{4R D(X)} \epsilon_{ij} dX^i \wedge dX^j \\ &\equiv \frac{1}{2} B_{AB} dX^A \wedge dX^B. \end{aligned} \tag{41}$$

This has  $SO(3) \otimes SO(2)$  symmetry. The components subject to the condition of tangential projection  $B_{AB} X^A = 0$  are

$$\begin{aligned} B_{ab} &= \frac{n_2}{2D(X)^3} \epsilon_{abc} X^c, & B_{ij} &= \frac{n_1}{2RD(X)} \epsilon_{ij}, \\ B_{ai} &= -B_{ia} = \frac{n_1}{2RD(X)^3} \epsilon_{ij} X^j X^a. \end{aligned} \quad (42)$$

These components are singular on  $\mathcal{C}$ .

The inverse matrix  $\alpha^{AB}$  which satisfies  $\alpha^{AB} B_{BC} = \delta_{AC} - X^A X^C / R^2$  is then given by

$$\begin{aligned} \alpha^{ab} &= -\frac{2D(X)}{n_2} \epsilon_{abc} X^c, & \alpha^{ij} &= -\frac{2D(X)^3}{n_1 R} \epsilon_{ij}, \\ \alpha^{ai} &= -\alpha^{ia} = -\frac{2D(X)}{n_1 R} \epsilon_{ij} X^j X^a. \end{aligned} \quad (43)$$

Now  $\alpha^{AB}$  defines the Poisson bracket.

$$\{F(X), G(X)\}_{PB} \equiv \frac{1}{2} \alpha^{AB} \partial_A F(X) \partial_B G(X) \quad (44)$$

We can easily show that  $\alpha^{AB}$  satisfies the condition of associativity.

$$\alpha^{AB} \partial_B (\alpha^{CD} / R^2) + (\text{cyclic permutations in } A, C, D) = 0 \quad (45)$$

This is also valid on the circle (20), because  $\alpha^{AB} = \partial_A \alpha^{BC} = 0$  on  $\mathcal{C}$ . In terms of  $\alpha^{AB}$  we can define the non-commutative product.[16]

$$\begin{aligned} &F(X) \star' G(X) \\ &= F(X)G(X) + i \alpha^{AB} \partial_A F(X) \partial_B G(X) \\ &\quad - \frac{1}{2} \alpha^{AB} \alpha^{CD} \partial_A \partial_C F(X) \partial_B \partial_D G(X) \\ &\quad - \frac{1}{3} \alpha^{AB} (\partial_B \alpha^{CD}) \{ \partial_A \partial_C F(X) \partial_D G(X) - \partial_C F(X) \partial_A \partial_D G(X) \} \\ &\quad + \dots \end{aligned} \quad (46)$$

The drawback of this approach is that it is difficult to obtain an explicit expression of the non-commutative product in a closed form. From higher order terms symmetric terms like  $S^{AB} \partial_A F \partial_B G$  ( $S^{BA} = S^{AB}$ ) will appear and even the coefficient function of the anti-symmetric term,  $\alpha^{AB}(X) \partial_A F(X) \partial_B G(X)$ , will be modified. In view of the symmetry of  $B$  we, however, expect this product to be related to the Matrix configuration (7) and the product (12) obtained in sec.3. To establish this connection more investigation will be necessary. Because there are two integers,  $n_1, n_2$ , in (41), we suspect there will be more representations other than (7). Matrix configuration (7) will correspond to  $n_1 = 1$  and  $n_2 = 2j$ .

## 6 Discussions

In this paper we have explicitly constructed Matrix 4-brane configurations (7) and then defined the non-commutative  $S^4$  algebra (8) and Matrix theory action (10). It turned out that the algebra is not invariant under  $SO(5)$  but only under  $SO(3) \otimes SO(2)$  and that the algebra is not even a Lie algebra. These Matrix configurations take the forms of tensor products of  $2 \times 2$  matrix and  $(2j+1) \times (2j+1)$  one. ( $j = 1/2, 1, 3/2, \dots$ ) Then it is natural to expect more representations which depend on a parameter other than  $j$ . Actually, by simply replacing the Pauli matrices  $\sigma_a$  in (7) by  $2 T_{(j')}^a$  we obtain

$$\hat{X}_0^a = \frac{\alpha}{j'(j'+1)} T_{(j')}^3 \otimes T_{(j)}^a, \quad \hat{X}_0^i = \frac{\beta}{2j'(j'+1)} T_{(j')}^{i-3} \otimes \mathbf{1}_{2j+1}, \quad (47)$$

( $a = 1, 2, 3, \quad i = 4, 5$ ). While we still have  $SO(3) \otimes SO(2)$  symmetry, it turns out that the algebra must be modified and the RHS of the algebra is not polynomials.

$$\begin{aligned} [\hat{X}^a, \hat{X}^b] &= \frac{2\alpha}{\beta^2} j'(j'+1) \epsilon_{abc} \left( \hat{X}^c [\hat{X}^4, \hat{X}^5] + [\hat{X}^4, \hat{X}^5] \hat{X}^c \right), \\ [\hat{X}^a, \hat{X}^i] &= \frac{j'(j'+1)}{\alpha} \epsilon_{abc} \epsilon_{ij} \left( \hat{X}^b [\hat{X}^c, \hat{X}^j] - [\hat{X}^b, \hat{X}^j] \hat{X}^c \right), \\ [\hat{X}^4, \hat{X}^5] &= \frac{i\beta^2}{4\alpha j'(j'+1)} \left( \frac{-i}{j(j+1)} \epsilon_{abc} \hat{X}^a \hat{X}^b \hat{X}^c \right)^{1/3} \end{aligned} \quad (48)$$

Here the cubic root of a hermitian matrix is defined such that the resultant matrix is also hermitian. Although apparently there is no problem in the existence of the cubic root itself, more elaborate extension of (7) may be needed. Nonetheless, the construction of the non-commutative product  $\star$  in sec.3 can also be carried out for the configurations (47) straightforwardly. Moreover we expect that the corresponding values of integers in the  $B$  field (41) are given by  $n_1 = 2j'$  and  $n_2 = 2j$ .

We also constructed a non-commutative product on  $S^4$  which corresponds to the Matrix configuration (7). Because this configuration has the form of the tensor product of two matrices, we made these matrices to correspond to two  $S^2$ 's, and then directly constructed the non-commutative product  $\star$  on  $S^4$  in terms of that on  $S^2$ . We found that the manifold corresponding to the configuration (7) is a twofold  $S^4$ .

Here we should stress that to obtain a non-commutative product on  $S^4$  we can neglect this doubling structure. The product (24) restricted to one of the

doubled  $S^4$  provides a full-fledged non-commutative product on  $S^4$ . In this case, however, the connection to the Matrix theory may be lost.

We also noticed that the product and the functions on  $S^4$  have singularity (indeterminateness) on the circle (20). The image of a point on this circle is  $S^2$ . In spite of this singularity we can construct a non-commutative product and a finite action integral. We then worked out an explicit expression for the non-commutative product for the representation  $j = 1/2$ .

Let us now consider the commutative limit of the algebra of (47). We expect that in the suitable large  $j, j'$  limit the commutative geometry of  $S^4$  is recovered. We will show this. When the constants  $\alpha, \beta$  are related by  $\beta = 2\alpha\sqrt{j(j+1)}$ , the matrices  $\hat{X}_0^A$  satisfy the constraint.

$$(\hat{X}_0^A)^2 = \hat{R}^2, \quad \hat{R} = \alpha \sqrt{\frac{j(j+1)}{j'(j'+1)}} \quad (49)$$

To make the radius  $\hat{R}$  finite we will keep  $\alpha$  and the ratio  $j/j'$  fixed. The commutators of  $\hat{X}_0^A$  are given by

$$\begin{aligned} [\hat{X}_0^a, \hat{X}_0^b] &= i \frac{\alpha^2}{j'^2(j'+1)^2} \epsilon_{abc} (T_{(j')}^3)^2 \otimes T_{(j)}^c, \\ [\hat{X}_0^a, \hat{X}_0^i] &= i \frac{\alpha\beta}{2j'^2(j'+1)^2} \epsilon_{ij} T_{(j')}^{j-3} \otimes T_{(j)}^a, \\ [\hat{X}_0^4, \hat{X}_0^5] &= i \frac{\beta^2}{4j'^2(j'+1)^2} T_{(j')}^3 \otimes \mathbf{1}_{2j+1} \end{aligned} \quad (50)$$

( $a, b, c = 1, 2, 3, \quad i, j = 4, 5$ ) Since the matrix elements of  $T_{(j)}^a$  and  $T_{(j')}^a$  are at most order  $\mathcal{O}(j)$  and  $\mathcal{O}(j')$ , respectively, all the commutators vanish like  $1/j$  in the above mentioned  $j, j' \rightarrow \infty$  limit. Therefore this limit yields a commutative  $S^4$  with a finite radius  $\hat{R}$ .

As for the next investigation we are planning to study the following subjects. Our Matrix configuration (7) turned out to correspond to a twofold  $S^4$ . We will study this structure in more details and seek for the possibility to construct new Matrix configurations which do not lead to a doubling structure. We are also planning to extend our construction of Matrix theory configuration to higher dimensional even sphere  $S^{2m}$ . Finally, we have not discussed a supersymmetric extension of our Matrix theory action (10). For this purpose we will need to introduce extra coordinates  $X^6, \dots, X^9$  in addition to fermions.

## Acknowledgments

The work of R.N. is supported in part by Grant-in-Aid (No.13135201) from the Ministry of Education, Science, Sports and Culture of Japan (Priority Area of Research (2)).

## Appendix A: Multiplication Rule of functions on $(S^4)_{1/2}$

$$\begin{aligned}
\frac{X^a}{R} \star \frac{X^b}{R} &= \delta_{ab} + i\epsilon_{abc} \frac{X^c}{D(X)}, & \frac{X^i}{R} \star \frac{X^j}{R} &= \delta_{ij} + i\epsilon_{ij} \frac{D(X)}{R}, \\
\frac{X^a}{R} \star \frac{X^i}{R} &= i\epsilon_{ij} \frac{X^j X^a}{RD(X)}, & \frac{X^i}{R} \star \frac{X^a}{R} &= -i\epsilon_{ij} \frac{X^j X^a}{RD(X)}, \\
\frac{X^a}{D(X)} \star \frac{D(X)}{R} &= \frac{X^a}{R}, & \frac{D(X)}{R} \star \frac{X^a}{D(X)} &= \frac{X^a}{R}, \\
\frac{X^i}{R} \star \frac{D(X)}{R} &= -i\epsilon_{ij} \frac{X^j}{R}, & \frac{D(X)}{R} \star \frac{X^i}{R} &= i\epsilon_{ij} \frac{X^j}{R}, \\
\frac{X^i}{R} \star \frac{X^j X^a}{RD(X)} &= \delta_{ij} \frac{X^a}{D(X)} + i\epsilon_{ij} \frac{X^a}{R}, \\
\frac{X^i X^a}{RD(X)} \star \frac{X^j}{R} &= \delta_{ij} \frac{X^a}{D(X)} + i\epsilon_{ij} \frac{X^a}{R}, \\
\frac{X^a}{D(X)} \star \frac{X^b}{D(X)} &= \delta_{ab} + i\epsilon_{abc} \frac{X^c}{D(X)}, \\
\frac{X^i X^a}{RD(X)} \star \frac{X^b}{D(X)} &= \delta_{ab} \frac{X^i}{R} + i\epsilon_{abc} \frac{X^i X^c}{RD(X)}, \\
\frac{X^a}{D(X)} \star \frac{X^i X^b}{RD(X)} &= \delta_{ab} \frac{X^i}{R} + i\epsilon_{abc} \frac{X^i X^c}{RD(X)}, \\
\frac{X^a}{R} \star \frac{X^b}{D(X)} &= \delta_{ab} \frac{D(X)}{R} + i\epsilon_{abc} \frac{X^c}{R}, \\
\frac{X^a}{D(X)} \star \frac{X^b}{R} &= \delta_{ab} \frac{D(X)}{R} + i\epsilon_{abc} \frac{X^c}{R}, \\
\frac{X^i X^a}{RD(X)} \star \frac{D(X)}{R} &= -i\epsilon_{ij} \frac{X^j X^a}{RD(X)}, \\
\frac{D(X)}{R} \star \frac{X^i X^a}{RD(X)} &= i\epsilon_{ij} \frac{X^j X^a}{RD(X)}, \\
\frac{X^a}{R} \star \frac{D(X)}{R} &= \frac{X^a}{D(X)}, & \frac{D(X)}{R} \star \frac{X^a}{R} &= \frac{X^a}{D(X)},
\end{aligned}$$

$$\begin{aligned}
\frac{X^a}{R} \star \frac{X^i X^b}{RD(X)} &= i\epsilon_{ij}\delta_{ab}\frac{X^j}{R} - \epsilon_{ij}\epsilon_{abc}\frac{X^j X^c}{RD(X)}, \\
\frac{X^i X^a}{RD(X)} \star \frac{X^b}{R} &= -i\epsilon_{ij}\delta_{ab}\frac{X^j}{R} + \epsilon_{ij}\epsilon_{abc}\frac{X^j X^c}{RD(X)}, \\
\frac{X^i X^a}{RD(X)} \star \frac{X^j X^b}{RD(X)} &= \delta_{ij}\delta_{ab} + i\delta_{ij}\epsilon_{abc}\frac{X^c}{D(X)} \\
&\quad + i\epsilon_{ij}\delta_{ab}\frac{D(X)}{R} - \epsilon_{ij}\epsilon_{abc}\frac{X^c}{R}, \\
\frac{D(X)}{R} \star \frac{D(X)}{R} &= 1
\end{aligned} \tag{51}$$

Here  $\epsilon_{abc}$  and  $\epsilon_{ij}$  are Levi-Civita symbols with  $\epsilon_{123} = \epsilon_{45} = +1$ .

## Appendix B: Coefficient Functions $L^{(n)}$

$$\begin{aligned}
L_{a,b}^{(1)} &= R^2\delta_{ab} - X^a X^b + iR^2\epsilon_{abc}\frac{X^c}{D(X)}, \\
L_{i,j}^{(2)} &= R^2\delta_{ij} - X^i X^j + iR^2\epsilon_{ij}\frac{D(X)}{R}, \\
L_{a,i}^{(3)} &= -X^a X^i + iR^2\epsilon_{ij}\frac{X^j X^a}{RD(X)}, \\
L_{i,a}^{(4)} &= -X^a X^i - iR^2\epsilon_{ij}\frac{X^j X^a}{RD(X)}, \\
L_{ab,c}^{(5)} &= \frac{R^2 - D(X)^2}{2}(X^a\delta_{bc} + X^b\delta_{ac}) - \frac{R^2 - D(X)^2}{D(X)^2}X^a X^b X^c \\
&\quad + i\frac{R^2 - D(X)^2}{2}(X^b\epsilon_{acd} + X^a\epsilon_{bcd})\frac{X^d}{D(X)}, \\
L_{a,bc}^{(6)} &= \frac{R^2 - D(X)^2}{2}(X^c\delta_{ab} + X^b\delta_{ac}) - \frac{R^2 - D(X)^2}{D(X)^2}X^a X^b X^c \\
&\quad + i\frac{R^2 - D(X)^2}{2}(X^c\epsilon_{abd} + X^b\epsilon_{acd})\frac{X^d}{D(X)}, \\
L_{ab,i}^{(7)} &= -\frac{1}{2}X^a X^b X^i + i\frac{R}{D(X)}\epsilon_{ij}X^j X^a X^b, \\
L_{a,bi}^{(8)} &= -D(X)^2 X^i\delta_{ab} + X^a X^b X^i - iD(X)\epsilon_{abc}X^c X^i + iRD(X)\delta_{ab}\epsilon_{ij}X^j \\
&\quad - i\frac{R}{D(X)}\epsilon_{ij}X^j X^a X^b - R\epsilon_{ij}\epsilon_{abc}X^j X^c, \\
L_{ai,b}^{(9)} &= -D(X)^2 X^i\delta_{ab} + X^a X^b X^i - iD(X)\epsilon_{abc}X^c X^i - iRD(X)\delta_{ab}\epsilon_{ij}X^j \\
&\quad + i\frac{R}{D(X)}\epsilon_{ij}X^j X^a X^b + R\epsilon_{ij}\epsilon_{abc}X^j X^c, \\
L_{i,ab}^{(10)} &= -\frac{1}{2}X^a X^b X^i - i\frac{R}{D(X)}\epsilon_{ij}X^j X^a X^b,
\end{aligned}$$

$$\begin{aligned}
L_{ai,j}^{(11)} &= \frac{D(X)^2}{2} X^a \delta_{ij} + i R^2 X^a \epsilon_{ij} \frac{D(X)}{R}, \\
L_{a,ij}^{(12)} &= -i \frac{R}{2} X^a (\epsilon_{ik} X^j + \epsilon_{jk} X^i) \frac{X^k}{D(X)}, \\
L_{i,aj}^{(13)} &= \frac{D(X)^2}{2} X^a \delta_{ij} + i R^2 X^a \epsilon_{ij} \frac{D(X)}{R}, \\
L_{ij,a}^{(14)} &= i \frac{R}{2} X^a (\epsilon_{ik} X^j + \epsilon_{jk} X^i) \frac{X^k}{D(X)}, \\
L_{ij,k}^{(15)} &= -i \frac{RD(X)}{2} (\delta_{ik} \epsilon_{jl} + \delta_{jk} \epsilon_{il}) X^l + \frac{D(X)^2}{4} (\epsilon_{ik} \epsilon_{jl} + \epsilon_{il} \epsilon_{jk}) X^l, \\
L_{i,jk}^{(16)} &= i \frac{RD(X)}{2} (\delta_{ij} \epsilon_{kl} + \delta_{ik} \epsilon_{jl}) X^l - \frac{D(X)^2}{4} (\epsilon_{ij} \epsilon_{kl} + \epsilon_{ik} \epsilon_{jl}) X^l, \\
L_{ab,cd}^{(17)} &= \frac{D(X)^2}{4} (R^2 - D(X)^2) (\delta_{ac} \delta_{bd} + \delta_{ad} \delta_{bc}) + \frac{R^2 - D(X)^2}{2D(X)^2} X^a X^b X^c X^d \\
&\quad + i \frac{R^2 - D(X)^2}{4} (X^a X^c \epsilon_{bde} + X^a X^d \epsilon_{bce} + X^b X^d \epsilon_{ace} + X^b X^c \epsilon_{ade}) \frac{X^e}{D(X)} \\
&\quad - \frac{R^2 - D(X)^2}{4} (\epsilon_{ace} \epsilon_{bdf} + \epsilon_{ade} \epsilon_{bcf}) X^e X^f, \\
L_{ij,kl}^{(18)} &= -\frac{D(X)^4}{4} (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) + \frac{D(X)^4}{4} (\epsilon_{ik} \epsilon_{jl} + \epsilon_{il} \epsilon_{jk}), \\
L_{ab,ci}^{(19)} &= -\frac{D(X)^2}{2} X^i (X^b \delta_{ac} + X^a \delta_{bc}) - i \frac{D(X)}{2} X^i (X^b \epsilon_{acd} + X^a \epsilon_{bcd}) X^d \\
&\quad + i \frac{RD(X)}{2} (X^b \delta_{ac} + X^a \delta_{bc}) \epsilon_{ij} X^j - \frac{R}{2} (X^b \epsilon_{acd} + X^a \epsilon_{bcd}) \epsilon_{ij} X^j X^d, \\
L_{ai,bc}^{(20)} &= -\frac{D(X)^2}{2} X^i (X^c \delta_{ab} + X^b \delta_{ac}) - i \frac{D(X)}{2} X^i (X^c \epsilon_{abd} + X^b \epsilon_{acd}) X^d \\
&\quad - i \frac{RD(X)}{2} (X^c \delta_{ab} + X^b \delta_{ac}) \epsilon_{ij} X^j + \frac{R}{2} (X^c \epsilon_{abd} + X^b \epsilon_{acd}) \epsilon_{ij} X^j X^d, \\
L_{ab,ij}^{(21)} &= -\frac{1}{2} \epsilon_{ik} \epsilon_{jl} X^k X^l X^a X^b, \\
L_{ij,ab}^{(22)} &= -\frac{1}{2} \epsilon_{ik} \epsilon_{jl} X^k X^l X^a X^b, \\
L_{ai,bj}^{(23)} &= R^2 D(X)^2 \delta_{ij} \delta_{ab} - D(X)^2 X^i X^j \delta_{ab} - \frac{R^2 - D(X)^2}{2} X^a X^b \delta_{ij} \\
&\quad + i R^2 D(X) \delta_{ij} \epsilon_{abc} X^c - i D(X) X^i X^j \epsilon_{abc} X^c \\
&\quad + i R^2 D(X)^2 \delta_{ab} \epsilon_{ij} \frac{D(X)}{R} - R D(X)^2 \epsilon_{ij} \epsilon_{abc} X^c, \\
L_{ai,jk}^{(24)} &= \frac{D(X)^2}{4} X^a (X^k \delta_{ij} + X^j \delta_{ik}), \\
L_{ij,ak}^{(25)} &= \frac{D(X)^2}{4} X^a (X^j \delta_{ik} + X^i \delta_{jk})
\end{aligned} \tag{52}$$

## References

- [1] J. Castellino, S. Lee and W. Taylor IV, *Longitudinal 5-branes as 4-spheres in Matrix theory*, Nucl.Phys. **B526** (1998) 334-350, hep-th/9712105.
- [2] T. Banks, W. Fischler, S. H. Shenker and L. Suskind, *M theory as a matrix model: a conjecture*, Phys.Rev. **D55** (1997) 5112-5128, hep-th/9610043.
- [3] N. Ishibashi, H. Kawai, Y. Kitazawa and A. Tsuchiya, *A Large N reduced model as superstring*, Nucl. Phys. **B498** (1997) 467-491, hep-th/9612115.
- [4] H. Grosse, C. Klimčik and P. Prešnajder, *On Finite 4D Quantum Field Theory in Non-Commutative Geometry*, Commun.Math.Phys. **180** (1996) 429-438, hep-th/9602115.
- [5] J. Medina and D. O'Connor, *Scalar field Theory on fuzzy  $S^4$* , JHEP **0311** (2003) 051, hep-th/0212170.
- [6] P. M. Ho and S. Rangoolam, *Higher dimensional geometries from matrix brane constructions*, Nucl.Phys. **B627** (2002) 266-288, hep-th/0111278.
- [7] S. Rangoolam, *On spherical harmonics for fuzzy spheres in diverse dimensions*, Nucl.Phys. **B610** (2001) 461-488, hep-th/0105006; *Higher dimensional geometries related Fuzzy odd-dimensional spheres*, JHEP **0210** (2002) 064, hep-th/0207111.
- [8] C. N. Yang, *Generalization of Dirac's monopole to  $SU_2$  gauge fields*, J.Math.Phys. **19** (1978) 320.
- [9] S. -C. Zhang and J. P. Hu, *A four-dimensional generalization of the quantum Hall effect*, Science **294** (2001) 823.
- [10] J. Madore, *Class. Quantum Grav.* **9** (1992) 69.
- [11] P. Prešnajder, *The origin of chiral anomaly and the noncommutative geometry*, J.Math.Phys. **41** (2000) 2789-2804, hep-th/9912050.
- [12] K. Hayasaka, R. Nakayama and Y. Takaya, *A new noncommutative product on the fuzzy two-sphere corresponding to the unitary representation of  $SU(2)$  and the Seiberg-Witten map*, Phys. Lett. **B553** (2003) 109-118, hep-th/0209240.

- [13] D. Berenstein, J. Maldacena and H. Nastase, *Strings in flat space and pp waves from  $\mathcal{N} = 4$  Super Yang Mills*, JHEP **0204** (2002) 013, hep-th/0202021.
- [14] R. C. Meyers, *Dielectric Branes*, JHEP **9912** (1999) 022, hep-th/9910053.
- [15] N. Seiberg and E. Witten, *String Theory and Noncommutative Geometry*, JHEP **9909** (1999) 032, hep-th/9908142.
- [16] M. Kontsevich, *Deformation quantization of Poisson manifolds*, q-alg/9709040.